SOME UNIVERSAL GRAPHS

BY

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ABSTRACT

It is shown that various classes of graphs have universal elements. In particular, for each n the class of graphs omitting all paths of length n and the class of graphs omitting all circuits of length at least n possess universal elements in all infinite powers.

0. Introduction

In this paper we will consider the question raised by Rado ([R1], [R2]) of whether or not a class of graphs has a *universal* element in an infinite power, i.e., one which contains any other element of that cardinality as a subgraph. To describe the classes of graphs we will consider, and to pose our problem a little more accurately we need, some definitions. Rather than deal just with graphs we will give the definitions in terms of *relational structures*; i.e. structures of the form $\langle X, R_1, \ldots, R_n \rangle$ where each R_i is a relation on X (that is a subset of X^m for some m). Given a structure $\langle A, S_1, \ldots, S_n \rangle$ we say that $\langle X, R_1, \ldots, R_n \rangle$ *weakly omits* A if there is no $Y \subseteq X$ such that

$$\langle A, S_1, \ldots, S_n \rangle \cong \langle Y, R_1 \restriction Y, \ldots, R_n \restriction Y \rangle$$

($R_i \restriction Y$ is defined to be $R_i \cap Y^m$ where $R_i \subseteq X^m$). We say $\langle X, R_1, \ldots, R_n \rangle$ *omits* $\langle A, S_1, \ldots, S_n \rangle$ if there is no one to one function f from A to X so that for all i

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whenever $(a_1, \ldots, a_m) \in S_i$ then $(f(a_1), \ldots, f(a_m)) \in R_i$. A *graph* is a structure $\langle X, G \rangle$ where G is a binary relation which is symmetric and antireflexive. According to the above definitions, $\langle X, G \rangle$ weakly omits (omits) another graph $\langle A, H \rangle$ if $\langle A, H \rangle$ is not isomorphic to an induced subgraph (subgraph) of $\langle X, G \rangle$. We will often denote a graph simply by the letter G.

Given a class of structures K, an element $M \in K$ with cardinality κ is to be *strongly* (*weakly*) *universal* if M does not weakly omit (does not omit) any member of K which has cardinality at most κ. In particular, a member G in a class of graphs is called strongly (weakly) universal, if G contains every other member with at most $|G|$ vertices as an induced subgraph (as a subgraph). Weakly universal graphs are often called, for simplicity, *universal*.

A structure M is said to be *homogeneous* if any isomorphism between finite substructures of M can be extended to an automorphism of M. The best known examples of strongly universal graphs are also homogeneous. All the countable homogeneous graphs are known [LW]. Any such graph, G, is strongly universal in the class of graphs which weakly omit the same finite graphs as G. A class of structures defined as those structures which weakly omit some collection of finite structures possesses a countable strongly universal homogeneous member iff its finite elements form an *amalgamation class*, namely any two finite elements can be isomorphically embedded in a common finite extension in the class and if A_1, A_2 are finite elements in the class with a common substructure B then there is a finite extension of both in the class. Furthermore, in the above situation the countable strongly universal homogeneous element is unique up to isomorphism (see [F] for more details on amalgamation classes).

When the finite graphs in K do not form an amalgamation class there are several examples to show that universal graphs may not exist. Generalizing earlier results of Shelah [S1], Hajnal and Pach [HP], and N. G. deBruijn (unpublished), Komjáth and Pach [KP] proved among other results the following. Given cardinals $1 \leqq \alpha \leqq \beta \leqq \kappa$, $\alpha < \omega \leqq \kappa$, if the Generalized Continuum Hypothesis is assumed, then the class of all graphs omitting the complete bipartite graph $K_{\alpha\beta}$ possesses a universal element of cardinality κ, if and only if, (i) $\kappa > \omega$ or (ii) $\kappa = \omega$, $\alpha = 1$ and $\beta \leqq 3$. In fact in these cases a strongly universal element also exists. Another result asserts that there is no countable universal planar graph [P].

In this paper we shall show several classes of graphs which do not possess countable universal homogeneous graphs but do possess universal graphs. A *circuit of length n* in a graph is a sequence $x_0, x_1, \ldots, x_n = x_0$ so that for all

$i, j < n$ there is an edge between x_i and x_{i+1} and $x_i \neq x_j$. We will use the word *cycle* to indicate a sequence $x_0, x_1, \ldots, x_n$ so that for all $i, j < n$ there is an edge between x_i and x_{i+1} (i.e. we drop the requirement that no vertex be repeated). A *path of length n* is a sequence $x_0, \ldots, x_n$ where for $i, j \leqq n$ there is an edge between x_i and x_{i+1} and $x_i \neq x_j$.

In this paper we will show that, given any natural number s, the class of graphs omitting all odd circuits of length at most $2s + 1$ contains a countable strongly universal element. (Theorem 1.3). Our Theorems 1.12 and 1.14 state that the classes of all graphs omitting (i) paths of length s, and (ii) all circuits of length at least s, respectively, both have strongly universal members of any infinite cardinality. This answers a question raised in [HP].

For uncountable cardinals κ, the question of whether or not a strongly universal graph of cardinality κ exists is more a model-theoretic question than a graph-theoretic one. Suppose K is a class of graphs which is the class of models of a first order theory. A necessary condition that K have universal elements is that it have the *joint embedding property*, i.e. that any two elements of K can be isomorphically embedded in a common element. Assume that K satisfies the joint embedding property. There are various complete first order theories, T, so that T has a model which embeds every countable member of K. If some such theory, T has a universal model in power κ, then K has a universal element in power κ. (To see that this claim is true, let A be isomorphically embedded in a model of T of cardinality κ. This model can then be embedded in the universal model of T.) We may be guaranteed a universal model by cardinal arithmetic, e.g. CH implies that any complete first order theory has a universal (in fact saturated) model of cardinality $\aleph_1$. We can also be guaranteed universal models by model-theoretic properties. A complete first order theory with an infinite model will have a universal model in every infinite cardinal, including $\aleph_0$, if it is ω-stable; it will have universal models in every power greater than or equal to the continuum, if it is superstable; and it will have universal models in every cardinal κ satisfying $\kappa^{\aleph_0} = \kappa$, if it is stable. In [HMS] a criterion is given for a graph to have a superstable theory. ([PZ] contains a criterion for a graph to have a stable theory.) For cardinals greater than or equal to the continuum the existence of universal elements in the classes of graphs omitting (i) paths of length s, and (ii) all circuits of length at least s, respectively, follows from the criterion in [HMS]. It is possible to deduce from the structural information given here about these classes that every graph in the class has an ω-stable theory.

There are independence results for other classes of graphs. For many classes of graphs it is consistent that there is no universal graph of cardinality $\aleph_1$. In [S2], Shelah points out that if $\aleph_2$ Cohen reals are added to the universe then there is no universal graph (in the class of all graphs) of cardinality $\aleph_1$. In the same model of set theory there is no universal triangle-free graph of cardinality $\aleph_1$. In [S2], it is shown that it is consistent with $\neg$CH that there is a strongly universal graph of cardinality $\aleph_1$. Mekler [M] has extended this result to other classes of structures. It is still open, as far as we know, whether it is consistent with $\neg$CH that there is a strongly universal triangle-free graph of cardinality $\aleph_1$.

1. Universal graphs

The basic technique in our construction of universal graphs is to construct a class of structures in some larger language (i.e. other relations), show that the class of graphs we are interested in is the class of *reducts* of this other class to the language of graphs (i.e. the structures arising when the additional relations are deleted), and show that there are universal objects in this larger class. The simplest example we will consider is the class of graphs which omit odd cycles of length at most $2s+1$.

DEFINITION. A structure $\langle X, G, F_1, \ldots, F_s \rangle$ is an *s-structure* on X if $\langle X, G \rangle$ is a graph (i.e. G is a binary symmetric antireflexive relation on X) and the F_i are binary symmetric relations on X satisfying:

(a) if $\langle x, y \rangle, \langle y, z \rangle \in G$, then $\langle x, z \rangle \in F_1$;

(b) if $\langle x, y \rangle \in F_a, \langle y, z \rangle \in F_b$, then $\langle x, z \rangle \in F_{a+b}$;

(c) for all $a \leqq s$, $G \cap F_a = \varnothing$.

PROPOSITION 1.1. *For any graph $\langle X, G \rangle$, G does not contain an odd circuit of length at most $2s+1$ iff $\langle X, G \rangle$ is the reduct of an s-structure.*

PROOF. Let $H_a = \{\langle x, y \rangle$: there are $x = u_0 \ldots, u_{2a} = y$ so that $\langle u_i, u_{i+1} \rangle \in G$ for all $i < 2a\}$. The definition of an s-system implies that for all $a \leqq s$, $H_a \subseteq F_a$. Further $\langle X, G, H_a : a \leqq s \rangle$ satisfies (a) and (b) in the definition of an s-system. Suppose $\langle X, G \rangle$ has no odd circuit of length $\leqq 2s+1$. Then $\langle X, G, H_a : a \leqq s \rangle$ also satisfies (c) in the definition; to be more precise, if $\langle x, y \rangle \in F_a \cap G$ then G contains a cycle of length $2a+1$ and hence an odd circuit of length $\leqq 2a+1$. Also if $\langle X, G \rangle$ can be expanded into an s-system, then $\langle X, G, H_a : a \leqq s \rangle$ is an s-system. Suppose now that $x_1, x_2, \ldots, x_{2a+1}$ is a cycle and $a \leqq s$. Then $\langle x_1, x_a \rangle \in H_a \cap G$ in contradiction of (c). □

In order to establish that there is a strongly universal s-structure we will go further and show:

THEOREM 1.2. *The class of finite s-structures is an amalgamation class.*

PROOF. Let $\langle Y, G, F_1, \ldots, F_s \rangle$ and $\langle Z, G', F'_1, \ldots, F'_s \rangle$ be s-structures and let $X = Y \cap Z$. Recursively define H_a on $Y \cup Z$ by $H_1 = F_1 \cup F'_1$ for $a > 1$, let $\langle x, y \rangle \in H_a$ if either $\langle x, y \rangle \in F_a \cup F'_a$ or there are u, v and y such that $u + v = a$ and $\langle x, y \rangle \in H_u$ and $\langle y, z \rangle \in H_v$. We now must show that $\langle Y \cup Z, G_1 \cup G_2, H_1, \ldots, H_s \rangle$ is an s-structure. All that we have to do is verify (c) and that $H_a \restriction Y = F_a$, $H_a \restriction Z = F'_a$. In fact we only have to verify the latter condition since any possible contradiction to (c) must happen either in Y or Z. We will use an auxilliary hypothesis, namely that, if $\langle y, z \rangle \in H_a$, $y \in Y \setminus X$, $z \in Z \setminus X$, then there exists $x \in X$ so that $\langle y, x \rangle \in F_u$, $\langle x, z \rangle \in F'_v$, and $a = u + v$. Suppose now that $\langle x, y \rangle \in H_a$ and $x, y \in Y$. If $\langle x, y \rangle \notin F_a$ then there must be $z \in Z \setminus X$ such that $\langle x, z \rangle \in F_u$, $\langle z, y \rangle \in F'_v$ and $u + v = a$. By the induction there are $x', y' \in X$ so that: $\langle x, x' \rangle \in F_{u_1}$, $\langle x', z \rangle \in F_{u_2}$ and $u_1 + u_2 = u$; and $\langle y, y' \rangle \in F_{v_1}$, $\langle y', z \rangle \in F_{v_2}$ and $v_1 + v_2 = v$. But then $\langle x', y' \rangle \in F_{u_2 + v_2}$, $\langle x, y' \rangle \in F_{u_1 + u_2 + v_2} = F_{u + v_2}$. Finally $\langle x, y \rangle \in F_{u + v_2 + v_1} = F_a$.
The proof of the auxiliary hypothesis is similar. □

THEOREM 1.3. *For all s there is a strongly universal countable graph in the class of graphs omitting odd circuits of length at most* $2s + 1$.

PROOF. There is a countable strongly universal homogeneous s-structure. Given a countable graph which contains no circuits of length at most $2s + 1$, expand it to an s-structure and then embed it in the countable universal homogenous s-structure. □

In the case where $s = 1$, i.e. we are dealing with triangle-free graphs, then F_1 can always be taken to be $\{\langle a, b \rangle$: there is no edge from a to $b\}$. In this case our proof specialises to the usual proof that there is a universal homogeneous triangle-free graph. Since it is consistent that there does not exist a universal triangle-free graph of cardinality $\aleph_1$, Theorem 1.3 cannot be generalized to arbitrary cardinals even in the case $s = 1$.

In studying classes of graphs omitting paths of a fixed length or circuits of length at most k, we use the notion of a graph-tree.

DEFINITION. A *graph-tree* is a structure $\langle X, G, \leqq \rangle$ where $\langle X, G \rangle$ is a graph, $\leqq$ is a pre-partial-ordering (i.e. reflexive and transitive) with the following properties:

(i) $\langle X/\equiv, \leqq \rangle$ is a tree where $x \equiv y$ iff $x \leqq y$ and $y \leqq x$. (A *tree* is a partial order such that the predecessors of any element is linearly ordered.)

(ii) There is a natural number n so that any strictly increasing sequence (in $\leqq$) has length at most n.

(iii) If $\langle x, y \rangle \in G$ then either $x \leqq y$ or $y \leqq x$.

We first discuss some operations on graph-trees. For the remainder of the discussion fix a graph-tree $\langle X, G, \leqq \rangle$. For any $a \in X$ let $[a, \infty)$ denote $\{x : a \leqq x\}$ and $(0, a) = \{x : x \leqq a$ but not $a \leqq x\}$. In words, $[a, \infty)$ is the union of the equivalence classes (with respect to $\equiv$) which are $\geqq a/\equiv$ and $(0, a)$ is the union of the $\equiv$-classes which are $< a/\equiv$.

Proposition 1.4. *Suppose* $(0, a) = (0, b)$. *There is an automorphism of* $\langle X, G, \leqq \rangle$ *taking* $[a, \infty)$ *to* $[b, \infty)$ *and fixing* $X \setminus ([a, \infty) \cup [b, \infty))$ *iff* $[a, \infty)$ *is isomorphic to* $[b, \infty)$ *over* $(0, b)$ (*i.e. there is a one-to-one onto mapping* $\psi : (0, b) \cup [a, \infty) \to (0, b) \cup [b, \infty)$ *which fixes* $(0, b)$ *and preserves the induced graph-tree structure*).

Proof. If ψ is as above then the required automorphism, φ, is ψ on $[a, \infty)$, ψ^{-1} on $[b, \infty)$ and the identity elsewhere. The map φ obviously preserves $\leqq$. Using the fact that xGy implies that either $x \leqq y$ of $y \leqq x$, it is easy to verify that φ is an automorphism. The other direction of the proposition is trivial. □

Let $\langle T, \leqq \rangle$ be the tree structure on $X/\equiv \cup \{t_0\}$ obtained by adding an element t_0 to $X/\equiv$ and extending the ordering on $X/\equiv$ by letting t_0 be the minimum element of T. Call T the *tree attached to* X. We define the *rank* of an element t of T to be the maximum m so that there is $t = s_0 < \cdots < s_m$. Since there is a finite absolute bound on the length of chains in $\langle X, \leqq \rangle$, rank is well defined. For $t \in T$ we let define the *type* of t to be the isomorphism type of $[a, \infty)$ over $(0, a)$ if $t = a/\equiv$ for $a \in X$. In order to simplify the notation we will often use X in place of $\langle X, G, \leqq \rangle$ or $\langle X, G \rangle$. The meaning of X should be clear from the context. We now define two operations on graph-trees.

Definition. Suppose $\langle X, G, \leqq \rangle$ is a graph-tree and T is the tree attached to X. We define $S_k(X)$ by induction on rank. Rather than working with X directly, we will define the operation on the tree T. To begin, let $T_0 = T$ and $X_0 = X$. Suppose now that T_i and X_i have already been defined for some i less than the rank of t_0. Consider an element $t \in T$ of rank $i = 1$. Partition the immediate successors of t by letting t_1 and t_2 be in the same block iff they have the same type over t (in X_i). Now for each $t \in T$ of rank $i+1$, choose P_t a subset of the immediate successors of t so that any block B in the partition $|B \cap P_t| = \min\{|B|, k\}$. Let T_{i+1} be the tree consisting of those elements of T_i of rank $\geqq i+1$ together with the $\{s$: for some $t \in T_i$ of rank $i+1$ there is $t_1 \in P_t$ so that $t_1 \leqq s\}$. Let

$$X_{i+1} = \bigcup\{t : t \in T_{i+1} \text{ and } t \neq t_0\},$$

where X_{i+1} is given the induced graph-tree structure. If n is the rank of t_0 then let $S_k(X) = X_n$.

The following proposition is immediate from the construction.

Proposition 1.5. *For all natural numbers n and k, there is a natural number $m = m(n, k)$ with the property that if $\langle X, G, \leqq \rangle$ is a graph-tree where every strictly increasing sequence in $\leqq$ is of length $\leqq n$ and every equivalence class has cardinality $\leqq n$, then $|S_k(X)| \leqq m$.*

After shrinking a graph-tree, we will want to expand it.

Definition. Let $\langle X, G, \leqq \rangle$ be a graph-tree and T the attached tree. Suppose $t_1 = a/\equiv$ an immediate successor of t (in T) and κ is a cardinal. By adding κ copies of t_1 at t, we mean that κ disjoint copies of $[a, \infty)$ are added to T and each copy is isomorphic to $[a, \infty)$ over $(0, a)$. Given a cardinal κ and a natural number k, define $E_k^\kappa(X)$ as follows. Let n be the maximum height of the elements of T. Let $X_0 = X$ and $T_0 = T$. Suppose X_i and T_i have already been defined for some $i < n$. For each t of height i and t_1, an immediate successor of t such that there are at least k immediate successors of t with the same type, attach κ copies of t_1 at t. Thus we form T_{i+1} and X_{i+1}. Let $E_k^\kappa(X) = X_n$.

The following proposition should be clear from the construction.

Proposition 1.6. *Let $k < \omega$, X a graph-tree and κ an infinite cardinal so that $|X| \leqq \kappa$.*

(1) *For any $m < k$ elements of X and $k - m$ elements of $S_k(X)$ there is an automorphism of a sub-graph-tree of X which carries the m elements of X to $S_k(X)$ and fixes the $k - m$ elements of $S_k(X)$.*

(2) *For any k elements of $E_k^\kappa(X)$ there is an automorphism of $E_k^\kappa(X)$ which takes these elements to X.*

(3) *X is an induced substructure of $E_k^\kappa(S_k(X))$.*

PROPOSITION 1.7. *If (X, G) omits paths of length k, then so does $E_k^\kappa(S_k(X))$.*

PROOF. Clearly, $S_k(X)$ does not contain a path of length k. Since any k elements of $E_k^\kappa(S_k(X))$ can be carried by an automorphism to $S_k(X)$, they cannot form a path of length k. □

Recall that a graph is *2-connected* if any 2 distinct points lie on a circuit. We will study graphs which omit all circuits of length $\geqq k$, by studying 2-connected graphs.

PROPOSITION 1.8. *Suppose that κ is an infinite cardinal. If $\langle X, G, \leqq \rangle$ is a graph-tree, X is 2-connected and omits circuits of length at least k, then $E_k^\kappa(S_kX))$ is 2-connected.*

PROOF. First suppose $x_1, x_2 \in S_k(X)$. Let C be a circuit in X which contains x_1 and x_2. Since $|C| < k$, there is an automorphism of a sub-graph-tree of X which fixes x_1, x_2 and carries C to $S_k(X)$. Now consider any 2-connected X which omits circuits of length at least k. Suppose $x_1, x_2 \in E_k^\kappa(X)$. Choose an automorphism φ which carries x_1, x_2 to X. Let C be a circuit in X which contains $\varphi(x_1)$ and $\varphi(x_2)$. Then $\varphi^{-1}(C)$ is a circuit containing x_1 and x_2. □

Next we analyse connected graphs which omit paths of a fixed length.

PROPOSITION 1.9. *Suppose $\langle X, G \rangle$ is a connected graph with no paths of length $n+1$. Then any two paths of length n have a common vertex.*

PROOF. Assume not. Let $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$ be two disjoint paths of length n. Since $\langle X, G \rangle$ is connected there is a path $x_i = u_0 \ldots, u_m = y_j$, where x_i and y_j are the only elements this path has in common with $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. By listing the elements in reverse order, if necessary, we can assume $j \leqq n/2 \leqq i$. Hence $x_1, \ldots, x_i = u_0, \ldots, u_m = y_j, \ldots, y_n$ is a path of length at least $n+1$. □

PROPOSITION 1.10. *If $\langle X, G \rangle$ is a connected graph omitting paths of length k, then there is a pre-partial ordering $\leqq$ so that $\langle X, G, \leqq \rangle$ is a graph-tree where each equivalence class has cardinality at most k and every strictly ascending sequence has cardinality at most k.*

PROOF. We prove the theorem by induction on k. If $k = 1$, then any connected component has cardinality 1, so there is nothing to prove. Assume now the theorem is true for $k = n$, and $\langle X, G \rangle$ is a connected graph which omits paths of length $n+1$. If $\langle X, G \rangle$ also omits paths of length n, we are done by the induction hypothesis. So assume $x_0, \ldots, x_n$ is a path of length n. Let $\{C_i : i \in I\}$ enumerate the connected components of $X \setminus \{x_0, \ldots, x_n\}$. By Proposition 1.9 each C_i omits paths of length n, so, by the induction hypothesis, there is a pre-partial ordering $\leqq_i$ of C_i with the property that $\langle C_i, G \restriction C_i, \leqq_i \rangle$ is a graph-tree where each equivalence class has cardinality at most n and every strictly ascending sequence in $\leqq_i$ has cardinality at most n. Now define $x \leqq y$ iff either $x = x_m$ for some $0 \leqq m \leqq n$ or $x, y \in C_i$ and $x \leqq_i y$ for some $i \in I$. □

Lemma 1.11. *For any $k < \omega$ and cardinal κ there is a finite set of connected graphs of cardinality κ each of which omits paths of length k so that any connected graph of cardinality $\leqq \kappa$ can be embedded as an induced subgraph in one of these.*

Proof. Suppose that $\langle X, G \rangle$ is a graph which omits paths of length k. Choose $\leqq$ so that $\langle X, G, \leqq \rangle$ is a graph-tree as in Proposition 1.10. Then there is a number n (which can be chosen independently of $\langle X, G \rangle$ — see Proposition 1.5) so that $|S_k(X)| \leqq n$. Thus we can take as our family of graphs the reduct of $\{E_k^\kappa(X) : X$ is a graph-tree of cardinality at most n which omits paths of length $k\}$ to the language of graphs. □

Theorem 1.12. *For any cardinal κ and natural number k, there is a strongly universal graph in the class of graphs which omit paths of length k.*

Proof. The desired graph G_0 can be formed by taking the disjoint sum of κ copies of each of the graphs guaranteed by Lemma 1.10. Then given any graph G of size $\leqq \kappa$, which omits paths of length k, we can embed it in our graph by embedding each connected component of G into a connected component of G_0. □

For graphs which omit circuits of length at least k, we follow a similar strategy. Every graph can be divided into 2-connected components. The division of a connected graph into 2-connected pieces induces a tree structure on the pieces. Namely we can take any piece as our minimum node. A piece X is an immediate successor of Y if X is not $\leqq Y$ and either $X \cap Y$ is a singleton (in which case there is no edge from X to Y) or there is a (necessarily unique) edge from X to Y (in which case $X \cap Y$ is empty). We will show:

Lemma 1.13. *For any infinite cardinal κ and natural number k there is a finite family $\mathcal{F}$ of 2-connected graphs of size κ omitting all circuits of length at* *least k, so that any 2-connected of size κ omitting all circuits of length at least k can be embedded as an induced subgraph of a member of $\mathcal{F}$.*

Given this lemma we can prove the following:

THEOREM 1.14. *For any infinite cardinal κ and natural number k there is a strongly universal graph of size κ in the class of the graphs which omit circuits of length at least k.*

PROOF. It is enough to show that there is a graph which is universal among the connected ones. To accomplish this begin with any graph in the family $\mathcal{F}$, then attach to it in every possible way κ copies of each graph in $\mathcal{F}$ and continue this process at every step. □

It remains to prove Lemma 1.13.

PROPOSITION 1.15. *If $\langle X, G \rangle$ is a 2-connected graph which omits all circuits of length at least k, then $\langle X, G \rangle$ omits all paths of length k^2.*

PROOF. Let $x_0, x_2, \ldots, x_{k^2}$ be a path of length k^2. Since $\langle X, G \rangle$ is 2-connected there is a circuit containing x_0 and x_{k^2}. By our assumption, this circuit has at most $k-1$ elements. Hence there is a path $x_0 = y_0, \ldots, y_s = x_{k^2}$ with $s \leqq (k-1)/2$. So there is i and $j \geqq k$ and $r < t$ so that $x_i = y_r, x_{i+j} = y_t$ and if all $r < l < t$ and $m < j$ then $x_m \neq y_l$. But $x_i, y_{s+1}, \ldots, y_t, x_{i+t-1}, \ldots, x_i$ is a circuit of length $\geqq k$. □

PROOF (of Lemma 1.13). Choose a pre-partial ordering $\leqq$ so that $\langle X, G, \leqq \rangle$ is a graph-tree where any equivalence class has cardinality $< k^2$ and any strictly increasing sequence in $\leqq$ has cardinality $< k^2$. It is enough to show that $E^{\kappa}_{k^2} S_{k^2}(X)$ is 2-connected and omits all circuits of length at least k. Since X is 2-connected and has no circuits of length at least k, Lemma 1.8 shows that X is 2-connected. Since $E^{\kappa}_{k^2} S_{k^2}(X)$ omits paths of length k^2, it also omits all circuits of length $k^2 + 1$. Finally if $E^{\kappa}_{k^2} S_{k^2}(X)$ contains a circuit of length at most k^2, then, by Proposition 1.6, X contains a circuit of the same length. □

The results on graphs which omit paths of length k or circuits of length at least k hide structure theorems for graphs in these classes. Namely we can construct all such graphs by starting with a finite collection of graph-trees and then generate the other graphs by simple operations.

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